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On an inverse spectral problem for a quadratic Jacobi matrix pencil[☆]

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Abstract

Given two monic polynomials P_{2n} and P_{2n-2} of degree $2n$ and $2n-2$ ($n \geq 2$) with complex coefficients and with disjoint zero sets. We give necessary and sufficient conditions on these polynomials such that there exist two $n \times n$ Jacobi matrices B and C for which

$$P_{2n}(\lambda) = \det(\lambda^2 I_n + \lambda B + C), \quad P_{2n-2}(\lambda) = \det(\lambda^2 I_{n-1} + \lambda B_1 + C_1),$$

where B_1 and C_1 are the $(n-1) \times (n-1)$ Jacobi matrices obtained from B and C by deleting the last row and the last column. The zeros of P_{2n} and P_{2n-2} are the eigenvalues of the quadratic Jacobi matrix pencils on the right-hand side of the equalities, whence the title of the paper. The problem is formulated and solved in a slightly more general form.

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1. Introduction

In this paper we consider an inverse spectral problem for a quadratic matrix pencil Q of the form

$$Q(\lambda) = \lambda^2 I_n + \lambda B + C, \quad (1.1)$$

where I_n is the $n \times n$ identity matrix and B and C are $n \times n$ Jacobi (symmetric and tri-diagonal) matrices. The eigenvalues of Q are the zeros of the monic polynomial $\det Q$ of degree $2n$. We denote by Q_1 the quadratic pencil obtained from Q by deleting its last row and its last column and we call Q_1 the truncation of Q . The inverse spectral problem is the following:

Given two sequences of complex numbers $\{\lambda_j\}_{j=1}^{2n}$ and $\{\mu_k\}_{k=1}^{2n-2}$. Determine necessary and sufficient conditions on these numbers such that they are the eigenvalues of an $n \times n$ quadratic Jacobi matrix pencil Q of the form (1.1) and its truncation Q_1 , respectively.

The problem can also be formulated and will be solved in the following form:

Given two monic polynomials P_{2n} and P_{2n-2} of degree $2n$ and $2n - 2$, respectively. Determine necessary and sufficient conditions on these polynomials under which there is an $n \times n$ quadratic Jacobi matrix pencil Q of the form (1.1) such that

$$P_{2n} = \det Q, \quad P_{2n-2} = \det Q_1. \quad (1.2)$$

The monic polynomials on the left-hand side are determined by their $4n - 2$ zeros (taking into account their multiplicity) or equivalently, by their $4n - 2$ nonleading coefficients. Hence the equalities (1.2) determine $4n - 2$ nonlinear equations in the $4n - 2$ unknown entries of the Jacobi matrices B and C on the right-hand side. It is well known that such systems of equations may have no, finitely many, or a continuum of solutions, see, for example, [4, Corollary 2.6]. We show that the spectral inverse problem has only finitely many solutions (see Corollary 4.2), but we do not describe them. Instead we characterize the pair of polynomials $\{P_{2n}, P_{2n-2}\}$ for which the spectral inverse problem has *no* solution. See Theorem 6.1, which is the main result of this paper. We only assume that $\lambda_j \neq \mu_k$ or, which amounts to the same, that the zero sets of P_{2n} and P_{2n-2} are disjoint.

The quadratic pencil Q arises in the theory of vibrating systems, in this case from the system of n second-order differential equations

$$I_n x''(t) + Bx'(t) + Cx(t) = 0, \quad (1.3)$$

see, for example, the monograph [3] by S. Timoshenko, D.H. Young, and W. Weaver Jr. The matrices B and C are related to the damping and the stiffness of the vibrating system. Substituting $x(t) = ue^{\lambda t}$ in (1.3), we get the spectral equation $Q(\lambda)u = 0$. From this

equation a basis of solutions of the system (1.3) can be obtained. So the spectral inverse problem applied to a vibrating system is to find necessary and sufficient conditions on its spectral data and the spectral data of the truncated system which ensure that the damping and stiffness configuration is determined by Jacobi matrices.

The spectral inverse problem where the stiffness is assumed to be zero, $C = 0$, has been studied by H. Hochstadt [2], see also the monograph [7] by G.M.L. Gladwell and the recent survey paper [1] of M.T. Chu. This problem is linear in λ and concerns one Jacobi matrix. The quadratic spectral inverse problem for two Jacobi matrices, which we study here, is first considered by Y.M. Ram and S. Elhay [6], see also Y.M. Ram [5]. In [6], and in [1, Theorem 3.8] with reference to [6], it is stated that the inverse problem has a solution if the zeros of P_{2n} and P_{2n-2} have multiplicity 1 and the two zero sets are disjoint. But this cannot be true: see Example 6.2 below.

We briefly indicate the contents of this paper. In Section 2 we introduce the reduction method. We say the pair $\{P_{2n}, P_{2n-2}\}$ admits a 1-step reduction if it has a representation of the form

$$P_{2n}(\lambda) = (\lambda^2 + a\lambda + b)P_{2n-2}(\lambda) - (c\lambda + d)^2 P_{2n-4}(\lambda),$$

where $a, b, c, d \in \mathbb{C}$, $(c, d) \neq (0, 0)$, and P_{2n-4} is a monic polynomial of degree $2n - 4$. The pair $\{P_{2n-2}, P_{2n-4}\}$ is called a 1-step reduction of $\{P_{2n}, P_{2n-2}\}$ and we use the notation $\{P_{2n}, P_{2n-2}\} \rightarrow \{P_{2n-2}, P_{2n-4}\}$ to indicate this. We show that the spectral inverse problem is solvable if and only if there is a chain of 1-step reductions

$$\{P_{2n}, P_{2n-2}\} \rightarrow \{P_{2n-2}, P_{2n-4}\} \rightarrow \cdots \rightarrow \{P_{2j}, P_{2j-2}\} \rightarrow \cdots \rightarrow \{P_2, P_0\},$$

where each P_k is a monic polynomial of degree k and $P_0 \equiv 1$. The problem when a maximal chain exists can of course also be formulated for a pair of monic polynomials $\{P_{2n+1}, P_{2n-1}\}$ of odd degree, and in this sequel we consider both problems simultaneously. *Throughout the paper we assume that the zero sets of the two polynomials with which we start are disjoint.* In Section 3 we prove the uniqueness of a certain type of 1-step reductions, in Section 4 we give a criterion for when a pair of polynomials $\{P_n, P_{n-2}\}$ cannot be reduced at all, and in Section 5 we give a criterion for when $\{P_n, P_{n-2}\}$ has 1-step reductions and none of these can be reduced any further. Finally, in Section 6 we prove our main theorem, which is a criterion for when a pair $\{P_n, P_{n-2}\}$ does *not* admit a maximal chain of reductions, that is, a chain up to $\{P_2, P_0\}$ if n is even and up to $\{P_3, P_1\}$ if n is odd.

2. The reduction method

Recall that a square matrix $A = [a_{i,j}]$ with complex coefficients $a_{i,j}$ is called a Jacobi matrix if it is symmetric: $a_{i,j} = a_{j,i}$ and tri-diagonal: $a_{i,j} = 0$ if $|i - j| \geq 2$. Let B and C be two $n \times n$ Jacobi matrices with complex entries:

$$B = \begin{bmatrix} a_{n-1} & c_{n-2} & \cdots & 0 & 0 \\ c_{n-2} & a_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & c_0 \\ 0 & 0 & \cdots & c_0 & a_0 \end{bmatrix}, \quad C = \begin{bmatrix} b_{n-1} & d_{n-2} & \cdots & 0 & 0 \\ d_{n-2} & b_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_1 & d_0 \\ 0 & 0 & \cdots & d_0 & b_0 \end{bmatrix}.$$

Denote by B_j and C_j the $(n-j) \times (n-j)$ Jacobi matrices obtained from B and C , respectively, by deleting the last j rows and the last j columns, $j = 0, 1, \dots, n-1$; in particular, $B_0 = B$, $C_0 = C$, $B_{n-1} = a_{n-1}$, and $C_{n-1} = b_{n-1}$. Set $P_0 \equiv 1$ and

$$P_{2n-2j}(\lambda) = \det(\lambda^2 I_{n-j} + B_j \lambda + C_j), \quad j = 0, 1, \dots, n-1,$$

where I_k stands for the $k \times k$ identity matrix. Then

$$P_{2n}(\lambda) = (\lambda^2 + a_0 \lambda + b_0) P_{2n-2}(\lambda) - (c_0 \lambda + d_0)^2 P_{2n-4}(\lambda). \quad (2.1)$$

This equality shows the relation between the two polynomials P_{2n} and P_{2n-2} on the one hand and the entries in the last row and the last column of B and C and the coefficients of P_{2n-4} on the other hand. We can continue with the polynomials P_{2n-2} and P_{2n-4} :

$$P_{2n-2}(\lambda) = (\lambda^2 + a_1 \lambda + b_1) P_{2n-4}(\lambda) - (c_1 \lambda + d_1)^2 P_{2n-6}(\lambda),$$

which relates these two polynomials to the entries a_1 , b_1 , c_1 , and d_1 in the last row and column of B_1 and C_1 and the polynomial P_{2n-6} . In general, we have for $j = 0, 1, \dots, n-1$,

$$P_{2n-2j}(\lambda) = (\lambda^2 + a_j \lambda + b_j) P_{2n-2(j+1)}(\lambda) - (c_j \lambda + d_j)^2 P_{2n-2(j+2)}(\lambda), \quad (2.2)$$

where we have set $P_{-2} \equiv 0$.

These formulas indicate a reduction method to construct solutions of the inverse spectral problem mentioned in the Introduction, which we now explain. In the following we consider polynomials of arbitrary degree. As above, we use the convention to denote by P_k (sometimes also by \tilde{P}_k) a monic polynomial of degree equal to the subscript k . Let P_n and P_{n-2} be two polynomials. Then, by Euclid's algorithm, we have the unique representation

$$P_n(\lambda) = (\lambda^2 + \alpha \lambda + \beta) P_{n-2}(\lambda) + r(\lambda) \quad (2.3)$$

with $\alpha, \beta \in \mathbb{C}$ and remainder $r = P_n \pmod{P_{n-2}}$ which is a polynomial of degree $< n-2$. For $n \geq 4$ we look for representations of the following type:

$$P_n(\lambda) = (\lambda^2 + a \lambda + b) P_{n-2}(\lambda) - (c \lambda + d)^2 P_{n-4}(\lambda), \quad (c, d) \neq (0, 0). \quad (2.4)$$

If such a representation exists, we call the pair $\{P_n, P_{n-4}\}$ a 1-step reduction of the pair $\{P_n, P_{n-2}\}$. In this case we also say that $\{P_n, P_{n-2}\}$ admits a 1-step reduction $\{P_{n-2}, P_{n-4}\}$ and use the notation $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, P_{n-4}\}$. More generally, we say that the pair $\{P_n, P_{n-2}\}$ admits a k -step reduction if there is a chain

$$\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, P_{n-4}\} \rightarrow \dots \rightarrow \{P_{n-2k}, P_{n-2k-2}\},$$

where each subsequent pair is a 1-step reduction of the previous pair. The inverse problem is to find necessary and sufficient conditions on the pair $\{P_n, P_{n-2}\}$ such that it admits an $(\ell-1)$ -step reduction to $\{P_2, P_0\}$ if $n = 2\ell$ and to $\{P_3, P_1\}$ if $n = 2\ell + 1$, $\ell = 1, 2, \dots$. In this case we say that the inverse problem for the pair $\{P_n, P_{n-2}\}$ is solvable; otherwise it is called not solvable. The inverse problem for the pair $\{P_{2n}, P_{2n-2}\}$ of even degree coincides with the inverse spectral problem.

Proposition 2.1. *A pair $\{P_n, P_{n-2}\}$ admits a 1-step reduction (2.4) if and only if at least one of the following two conditions on the remainder term r in (2.3) holds:*

$$\deg(r) = n-4; \quad (2.5)$$

$$r - \gamma P_{n-2} \text{ has a zero of multiplicity } \geq 2 \text{ for some } \gamma \in \mathbb{C} \setminus \{0\}. \quad (2.6)$$

The proposition follows immediately from the definition. If (2.5) holds, we have $c = 0$ in (2.4), $-d^2 \neq 0$ is the leading coefficient of r , and $P_{n-4}(\lambda) = -r(\lambda)/d^2$; if (2.6) holds, we have $c^2 = \gamma \neq 0$ and $d = -c\lambda_0$, where λ_0 is a zero of multiplicity ≥ 2 of the function in (2.6).

Finally, we note that the coefficient α in (2.3) and the coefficient a in (2.4) are equal and given by

$$\alpha = a = -\sum_{j=1}^n \lambda_j + \sum_{k=1}^{n-2} \mu_k, \quad (2.7)$$

where $\{\lambda_j\}_{j=1}^n$ and $\{\mu_k\}_{k=1}^{n-2}$ are enumerations of the zeros of P_n and P_{n-2} , respectively, in which the number of repetitions of each zero is equal to its multiplicity.

3. The uniqueness of certain 1-step reductions

We denote the zero set of a polynomial p by

$$\text{nul}(p) = \{\mu \mid p(\mu) = 0\}.$$

Throughout the sequel we assume that the pair $\{P_n, P_{n-2}\}$ has the property

$$\text{nul}(P_n) \cap \text{nul}(P_{n-2}) = \emptyset. \quad (3.1)$$

It follows from (2.4) that every 1-step reduction and by induction every k -step reduction of $\{P_n, P_{n-2}\}$ has the same property. The following proposition concerns the uniqueness of a 1-step reduction $\{P_{n-2}, P_{n-4}\}$ of a special type, namely of the form

$$P_{n-4}(\lambda) = (\lambda - \lambda_0)^{n-4} + a_0, \quad (3.2)$$

$$P_{n-2}(\lambda) = (\lambda^2 + a_1\lambda + b_1)P_{n-4}(\lambda) + c_1 \quad (3.3)$$

with $\lambda_0, a_0, a_1, b_1, c_1 \in \mathbb{C}$ and $c_1 \neq 0$. The condition $c_1 \neq 0$ ensures that

$$\text{nul}(P_{n-2}) \cap \text{nul}(P_{n-4}) = \emptyset.$$

The proposition does not deal with the existence of a 1-step reduction of this type.

Proposition 3.1. *Assume that the polynomials P_n and P_{n-2} satisfy (3.1).*

- (i) *If $n \geq 9$, then there is at most one 1-step reduction of the form (3.2) and (3.3).*
- (ii) *If $n = 7, 8$, then there is at most one 1-step reduction of the form (3.2) and (3.3) with $a_0 = 0$.*
- (iii) *In the cases (i) and (ii), if the 1-step reduction $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, P_{n-4}\}$ of the form (3.2) and (3.3) exists, then P_n has the unique representation*

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)P_{n-2}(\lambda) - (c\lambda + d)^2P_{n-4}(\lambda)$$

except that the pair (c, d) may be replaced by the pair $(-c, -d)$.

Proof. Assume that a 1-step reduction $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, P_{n-4}\}$ exists which is of the form:

$$P_{n-4}(\lambda) = (\lambda - \lambda_0)^{n-4} + a_0, \quad (3.4)$$

$$P_{n-2}(\lambda) = (\lambda^2 + a_1\lambda + b_1)P_{n-4}(\lambda) + c_1, \quad (3.5)$$

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)P_{n-2}(\lambda) - (c\lambda + d)^2 P_{n-4}(\lambda), \quad (c, d) \neq (0, 0), \quad (3.6)$$

and that $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, \tilde{P}_{n-4}\}$ is a 1-step reduction of the same form:

$$\tilde{P}_{n-4}(\lambda) = (\lambda - \tilde{\lambda}_0)^{n-4} + \tilde{a}_0, \quad (3.7)$$

$$P_{n-2}(\lambda) = (\lambda^2 + \tilde{a}_1\lambda + \tilde{b}_1)\tilde{P}_{n-4}(\lambda) + \tilde{c}_1, \quad (3.8)$$

$$P_n(\lambda) = (\lambda^2 + \tilde{a}\lambda + \tilde{b})P_{n-2}(\lambda) - (\tilde{c}\lambda + \tilde{d})^2 \tilde{P}_{n-4}(\lambda), \quad (\tilde{c}, \tilde{d}) \neq (0, 0). \quad (3.9)$$

We first prove (iii). On account of (2.7), a in (3.6) and \tilde{a} in (3.9) are equal. Assume P_{n-4} in (3.4) and \tilde{P}_{n-4} in (3.7) coincide. Then from (3.6) and (3.9) and using the formulas (3.4) and (3.5), we obtain

$$\begin{aligned} & \gamma [(\lambda^2 + a_1\lambda + b_1)((\lambda - \lambda_0)^{n-4} + a_0) + c_1] \\ &= [(\tilde{c}\lambda + \tilde{d})^2 - (c\lambda + d)^2]((\lambda - \lambda_0)^{n-4} + a_0), \end{aligned} \quad (3.10)$$

where $\gamma = \tilde{b} - b$. For $n \geq 7$ this identity is equivalent to the system of two identities

$$\begin{cases} \gamma(\lambda^2 + a_1\lambda + b_1) - [(\tilde{c}\lambda + \tilde{d})^2 - (c\lambda + d)^2] = 0, \\ a_0\gamma(\lambda^2 + a_1\lambda + b_1) - a_0[(\tilde{c}\lambda + \tilde{d})^2 - (c\lambda + d)^2] + \gamma c_1 = 0. \end{cases}$$

This can be proved by expanding the two quadratic polynomials in curly brackets on both sides of (3.10) in terms of the three functions $(\lambda - \lambda_0)^2$, $\lambda - \lambda_0$, and 1. From the system it follows that $\gamma c_1 = 0$, hence $b = \tilde{b}$, because $c_1 \neq 0$. The first identity of the system now implies $(\tilde{c}\lambda + \tilde{d})^2 = (c\lambda + d)^2$, hence $(\tilde{c}, \tilde{d}) = \pm(c, d)$. Thus if $n \geq 7$ and $P_{n-4} = \tilde{P}_{n-4}$, then P_n has the unique representation; in particular, this holds for the cases (i) and (ii).

We now prove (ii). Assume $n \geq 7$ and $a_0 = \tilde{a}_0 = 0$. Then from (3.6) and (3.9) and using the formulas (3.4), (3.5), and (3.7), we obtain

$$\begin{aligned} & (\tilde{c}\lambda + \tilde{d})^2(\lambda - \tilde{\lambda}_0)^{n-4} \\ &= (c\lambda + d)^2(\lambda - \lambda_0)^{n-4} + \gamma[(\lambda^2 + a_1\lambda + b_1)(\lambda - \lambda_0)^{n-4} + c_1], \end{aligned} \quad (3.11)$$

where $\gamma = \tilde{b} - b$. Taking the derivative of the polynomials on both sides of this equality, we see that there is a polynomial q of degree ≤ 2 such that

$$(\tilde{c}\lambda + \tilde{d})[2\tilde{c}(\lambda - \tilde{\lambda}_0) + (n-4)(\tilde{c}\lambda + \tilde{d})](\lambda - \tilde{\lambda}_0)^{n-5} = q(\lambda)(\lambda - \lambda_0)^{n-5}. \quad (3.12)$$

Note that the factors on the left-hand side are not identically equal to 0 and hence $q \not\equiv 0$ also.

We claim that $\tilde{\lambda}_0 = \lambda_0$. If $n > 7$, this follows from counting the zeros of the polynomials on both sides of (3.12). If $n = 7$, then also $\tilde{\lambda}_0 = \lambda_0$. For if this equality does not hold, then

$$\lambda_0 \in \text{nul}(\tilde{c}\lambda + \tilde{d}) \cap \text{nul}(\tilde{c}(\lambda - \tilde{\lambda}_0)).$$

If $\tilde{c} \neq 0$, we obtain the contradiction $\tilde{\lambda}_0 = \lambda_0$; if $\tilde{c} = 0$, then $\tilde{d} = 0$ which contradicts the assumption $(\tilde{c}, \tilde{d}) \neq (0, 0)$. This proves the claim. From the claim, (3.4), (3.5), (3.7), and (3.8) we obtain the equality

$$(\lambda^2 + a_1\lambda + b_1)(\lambda - \lambda_0)^{n-4} + c_1 = (\lambda^2 + \tilde{a}_1\lambda + \tilde{b}_1)(\lambda - \lambda_0)^{n-4} + \tilde{c}_1,$$

which implies that $c_1 = \tilde{c}_1$, $a_1 = \tilde{a}_1$, and $b_1 = \tilde{b}_1$.

Now we consider (i). Assume $n \geq 9$ and $c_1 \neq 0$, $\tilde{c}_1 \neq 0$. From (3.6) and (3.9) using the formulas (3.4), (3.5), and (3.7), we obtain

$$\begin{aligned} (\tilde{c}\lambda + \tilde{d})^2[(\lambda - \tilde{\lambda}_0)^{n-4} + \tilde{a}_0] &= (c\lambda + d)^2[(\lambda - \lambda_0)^{n-4} + a_0] \\ &\quad + \gamma\{(\lambda^2 + a_1\lambda + b_1)[(\lambda - \lambda_0)^{n-4} + a_0] + c_1\}, \end{aligned} \quad (3.13)$$

where $\gamma = \tilde{b} - b$. We consider two cases: $n > 9$ and $n = 9$.

Case $n > 9$. By taking the third derivative of the polynomials on both sides of (3.13) and counting the zeros as in a similar case above, we find that $\tilde{\lambda}_0 = \lambda_0$. Without loss of generality we may assume that $\tilde{\lambda}_0 = \lambda_0 = 0$.

We claim that $a_0 = \tilde{a}_0$. If $\tilde{c} = c = 0$, the equality (3.13) implies that $\gamma = 0$, that is, $\tilde{b} = b$, and $\tilde{d}^2 = d^2$, $\tilde{a}_0 = a_0$, and the claim holds. If $(c, \tilde{c}) \neq (0, 0)$, we can assume without loss of generality that $\tilde{c} \neq 0$. Then (3.13) can be rewritten in the form:

$$\begin{aligned} (\tilde{c}\lambda + \tilde{d})^2(\lambda^{n-4} + \tilde{a}_0) &= (c\lambda + d)^2(\lambda^{n-4} + a_0) \\ &\quad + \gamma[(\lambda^2 + a_1\lambda + b_1)(\lambda^{n-4} + a_0) + c_1]. \end{aligned}$$

This identity is equivalent to the system of two identities

$$\begin{cases} (\tilde{c}\lambda + \tilde{d})^2 = (c\lambda + d)^2 + \gamma(\lambda^2 + a_1\lambda + b_1), \\ (\tilde{c}\lambda + \tilde{d})^2\tilde{a}_0 = (c\lambda + d)^2a_0 + \gamma(\lambda^2 + a_1\lambda + b_1)a_0 + \gamma c_1. \end{cases}$$

Multiply the first equality by a_0 and take the difference of both equalities to obtain $(\tilde{c}\lambda + \tilde{d})^2(\tilde{a}_0 - a_0) = \gamma c_1$. (This also holds when $a_0 = 0$.) Since $\tilde{c} \neq 0$, we have $\tilde{a}_0 = a_0$ (and $\gamma c_1 = 0$, but we do not need this here). This proves the claim. The claim, (3.4), and (3.7) show $P_{n-4} = \tilde{P}_{n-4}$. Now the equalities $a_1 = \tilde{a}_1$, $b_1 = \tilde{b}_1$, and $c_1 = \tilde{c}_1$ follow directly from (see (3.5) and (3.8)):

$$(\lambda^2 + a_1\lambda + b_1)P_{n-4}(\lambda) + c_1 = (\lambda^2 + \tilde{a}_1\lambda + \tilde{b}_1)P_{n-4}(\lambda) + \tilde{c}_1.$$

This completes the proof of (i) for this case.

Case $n = 9$. Suppose that $\lambda_0 \neq \tilde{\lambda}_0$. From (3.5) and (3.8) we get

$$P_7'''(\lambda) = 210s(\lambda)(\lambda - \lambda_0)^2 = 210\tilde{s}(\lambda)(\lambda - \tilde{\lambda}_0)^2, \quad (3.14)$$

where s and \tilde{s} are monic polynomials of degree 2. The second equality in (3.14) implies that $(\lambda - \tilde{\lambda}_0)^2$ is a divisor of s , that is, $s(\lambda) = (\lambda - \tilde{\lambda}_0)^2$, and hence

$$P_7'''(\lambda) = 210(\lambda - \tilde{\lambda}_0)^2(\lambda - \lambda_0)^2. \quad (3.15)$$

We rewrite (3.13) with $n = 9$ as

$$\tilde{q}(\lambda) = q(\lambda) + \gamma P_7(\lambda), \quad (3.16)$$

where

$$q(\lambda) = (c\lambda + d)^2[(\lambda - \lambda_0)^5 + a_0], \quad \tilde{q}(\lambda) = (\tilde{c}\lambda + \tilde{d})^2[(\lambda - \tilde{\lambda}_0)^5 + \tilde{a}_0].$$

We differentiate the functions on both sides of (3.16), and using (3.15) we get the following equality:

$$\tilde{t}(\lambda)(\lambda - \tilde{\lambda}_0)^2 = t(\lambda)(\lambda - \lambda_0)^2 + 210\gamma(\lambda - \tilde{\lambda}_0)^2(\lambda - \lambda_0)^2, \quad (3.17)$$

where, since $(c, d) \neq (0, 0)$ and $(\tilde{c}, \tilde{d}) \neq (0, 0)$, t and \tilde{t} are nonzero polynomials of degree ≤ 2 . We claim $\lambda_0 = \tilde{\lambda}_0$. To prove the claim, we assume $\lambda_0 \neq \tilde{\lambda}_0$ and show that $\lambda_0 = \tilde{\lambda}_0$. Assume $\lambda_0 \neq \tilde{\lambda}_0$. Then the identity implies that in fact $c \neq 0$, $\tilde{c} \neq 0$, and

$$t(\lambda) = 210c^2(\lambda - \lambda_0)^2, \quad \tilde{t}(\lambda) = 210\tilde{c}^2(\lambda - \tilde{\lambda}_0)^2,$$

that is, t and \tilde{t} have zeros of multiplicity 2. Now we make use of the following simple observation:

For $n \geq 3$, let $u(\lambda) = (\lambda - \mu)^2((\lambda - \lambda_0)^n + v)$ with $\mu, \lambda_0, v \in \mathbb{C}$. If $\mu \neq \lambda_0$, then $u'''(\lambda) = v(\lambda)(\lambda - \lambda_0)^{n-3}$, where

$$v(\lambda) = (n+2)(n+1)n(\lambda - \lambda_0)^2 - 2(n+1)n(n-1)(\lambda - \lambda_0)(\mu - \lambda_0) + n(n-1)(n-2)(\mu - \lambda_0)^2$$

has two distinct zeros. Thus, if $v(\lambda)$ has a zero of multiplicity 2, then $\mu = \lambda_0$.

This follows immediately from the inequality

$$(n+1)n(n-1) \neq \sqrt{(n+2)(n+1)n^2(n-1)(n-2)}.$$

This observation applied to $(u, v) = (q, t)$ and $(u, v) = (\tilde{q}, \tilde{t})$ yields

$$-c/d = \lambda_0, \quad -\tilde{d}/\tilde{c} = \tilde{\lambda}_0.$$

Substituting this into (3.17), we get

$$\tilde{c}^2(\lambda - \tilde{\lambda}_0)^4 = c^2(\lambda - \lambda_0)^4 + \gamma(\lambda - \tilde{\lambda}_0)^2(\lambda - \lambda_0)^2,$$

and this implies $\lambda_0 = \tilde{\lambda}_0$. Thus the claim is true. Now we can repeat the arguments used in the previous case to complete the proof. \square

4. A criterion when a 1-step reduction does not exist

For polynomials p and q with complex coefficients we define the parameter function

$$[p, q]_\gamma = p - \gamma q$$

with parameter $\gamma \in \mathbb{C}$, the Wronskian

$$\omega_{p,q} = pq' - p'q,$$

and the function

$$f_{p,q} = \frac{\omega_{p,q}}{\gcd(p, p') \gcd(q, q')},$$

where $\gcd(p, q)$ stands for the greatest common divisor of p and q , which by definition is a monic polynomial. All three functions are polynomials. We note the following properties:

- (a) $\omega_{p,q} \equiv 0$ if and only if $\alpha p + \beta q \equiv 0$ for some pair of complex numbers $(\alpha, \beta) \neq (0, 0)$.
- (b) $\mu \in \mathbb{C}$ is a zero of $\omega_{p,q}$, $\omega_{p,q} \not\equiv 0$, if and only if for some pair $(\alpha, \beta) \neq (0, 0)$, the polynomial $\alpha p + \beta q$ has a zero at μ of multiplicity ≥ 2 ; in this case, if $\text{nul}(p) \cap \text{nul}(q) = \emptyset$, then $\alpha \neq 0$ and $\beta \neq 0$.
- (c) If μ is a zero of p , then the quotient $p/\gcd(p, p')$ has a zero of multiplicity 1 at μ ; in particular, $(p'/\gcd(p, p'))(\mu) \neq 0$.
- (d) $\deg(f_{p,q}) = \deg(\omega_{p,q}) - \deg(\gcd(p, p')) - \deg(\gcd(q, q'))$.

Lemma 4.1. *Let p and q be nonconstant polynomials satisfying*

$$\text{nul}(p) \cap \text{nul}(q) = \emptyset. \quad (4.1)$$

Then there is a complex number $\gamma \neq 0$ such that the polynomial $[p, q]_\gamma$ has a zero of multiplicity ≥ 2 if and only if

$$\deg(f_{p,q}) > 0. \quad (4.2)$$

Moreover, if $\mu \in \text{nul}(f_{p,q})$, then $p(\mu) \neq 0$, $q(\mu) \neq 0$, and μ is a zero of multiplicity ≥ 2 of the polynomial $[p, q]_\gamma$ with parameter $\gamma = p(\mu)/q(\mu)$.

Proof. Assume $[p, q]_\gamma$ has a zero μ of multiplicity ≥ 2 and $\gamma \neq 0$. Then

$$(p - \gamma q)(\mu) = 0, \quad (p' - \gamma q')(\mu) = 0, \quad p(\mu) \neq 0, \quad q(\mu) \neq 0.$$

The first two equalities imply $\omega_{p,q}(\mu) = 0$ and the last two imply $\gcd(p, p')(\mu) \neq 0$ and $\gcd(q, q')(\mu) \neq 0$, and hence $f_{p,q}(\mu) = 0$. It follows that either $f_{p,q} \equiv 0$ or $\deg(f_{p,q}) > 0$. If $f_{p,q} \equiv 0$, then $\omega_{p,q} \equiv 0$ and therefore (see property (a) above) $\alpha p + \beta q \equiv 0$ for some pair of complex numbers $(\alpha, \beta) \neq (0, 0)$. Since p and q are nonconstant polynomials, we have $\text{nul}(p) = \text{nul}(q) \neq \emptyset$. This contradicts (4.1). Thus (4.2) prevails.

To prove the converse, assume (4.2). Then there is a $\mu \in \text{nul}(f_{p,q}) \cap \text{nul}(\omega_{p,q})$. We claim $\mu \notin \text{nul}(p) \cup \text{nul}(q)$. To prove the claim, we show that $p(\mu) = 0$ or $q(\mu) = 0$ leads to a contradiction. It suffices to consider the case $p(\mu) = 0$. Then, on account of (4.1), $q(\mu) \neq 0$ and from the identity

$$\frac{p'}{\gcd(p, p')} q = \frac{p}{\gcd(p, p')} q' + \gcd(q, q') f_{p,q}$$

it follows that $(p'/\gcd(p, p'))(\mu) = 0$. But this is in contradiction with property (c) above. This proves the claim. From the claim it follows that $p(\mu) \neq 0$ and $q(\mu) \neq 0$ and hence $\gamma := p(\mu)/q(\mu)$ is a well-defined nonzero complex number. Evidently, $[p, q]_\gamma(\mu) = 0$. That μ is a zero of multiplicity ≥ 2 of $[p, q]_\gamma$ follows from

$$[p, q]_\gamma'(\mu) = p'(\mu) - \gamma q'(\mu) = p'(\mu) - \frac{p(\mu)}{q(\mu)} q'(\mu) = -q(\mu) \omega_{p,q}(\mu) = 0. \quad \square$$

Corollary 4.2. *The inverse problem has only finitely many solutions.*

Proof. Set $r = P_n \pmod{P_{n-2}}$. We apply Proposition 2.1. If (2.5) holds, then $\{P_n, P_{n-2}\}$ has one 1-step reduction. If (2.6) holds, then the number of reductions is at most equal to $\deg(f_{r, P_{n-2}}) \leq n - 1$, according to Lemma 4.1 with $p = r$ and $q = P_{n-2}$. Hence $\{P_n, P_{n-2}\}$ has at most finitely many 1-step reductions. Since the sequence $\{P_n, P_{n-2}\} \rightarrow \dots \rightarrow \{P_3, P_1\}$ or $\{P_2, P_0\}$, depending on n being odd or even, is finite, the corollary holds true. \square

Corollary 4.3. *For $n = 4$ the inverse problem is solvable, that is, every pair $\{P_4, P_2\}$ with $\text{nul}(P_4) \cap \text{nul}(P_2) = \emptyset$ admits a 1-step reduction to $\{P_2, P_0\}$ with $P_0 \equiv 1$.*

Proof. We apply Proposition 2.1 with $r = P_4 \pmod{P_2}$. If $\deg(r) = 0$, then there is a reduction because (2.5) holds. Assume $\deg(r) = 1$. Then $\deg(\gcd(r, r')) = 0$, $\deg(\gcd(P_2, P'_2)) \leq 1$, $\deg(\omega_{r, P_2}) = 2$ and hence $\deg(f_{r, P_2}) \geq 1$. Now there is a reduction because, by Lemma 4.1, (2.6) holds. \square

Proposition 4.4. *The pair $\{P_n, P_{n-2}\}$ satisfying (3.1) admits no reduction if and only if the following conditions hold:*

$$n > 4; \quad (4.3)$$

$$P_{n-2}(\lambda) = (\lambda - \lambda_0)^{n-2} \quad \text{for some } \lambda_0; \quad (4.4)$$

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)P_{n-2}(\lambda) + c \quad \text{for some } c \neq 0. \quad (4.5)$$

Proof. Assume (4.3)–(4.5) hold. Then $r = P_n \pmod{P_{n-2}} \equiv c$. Since $n > 4$, (2.5) does not hold. From $\omega_{c, P_{n-2}}(\lambda) = c(n-2)(\lambda - \lambda_0)^{n-3}$,

$$\gcd(c, 0) = 1, \quad \gcd(P_{n-2}, P'_{n-2})(\lambda) = (\lambda - \lambda_0)^{n-3},$$

we obtain $f_{c, P_{n-2}} \equiv c(n-2)$. Hence, by Lemma 4.1, there is no $\gamma \in \mathbb{C} \setminus \{0\}$ such that $[c, P_{n-2}]_\gamma$ has a zero of multiplicity ≥ 2 , that is, (2.6) does not hold either. According to Proposition 2.1, the pair $\{P_n, P_{n-2}\}$ admits no 1-step reduction.

To prove the converse, assume $\{P_n, P_{n-2}\}$ has no 1-step reduction. Then, by Corollary 4.3, we have $n > 4$, which proves (4.3). Moreover, according to Proposition 2.1 and Lemma 4.1, we have that the polynomial $f_{r, P_{n-2}}$ is a constant τ , say. From (d) in the list of properties at the beginning of this section we have

$$\omega_{r, P_{n-2}} = \tau \gcd(r, r') \gcd(P_{n-2}, P'_{n-2}). \quad (4.6)$$

Assume $\deg(r) \neq 0$. By (4.6),

$$\deg(\omega_{r, P_{n-2}}) \leq \deg(r) - 1 + n - 3 = n - 4 + \deg(r),$$

whereas $\deg(\omega_{r, P_{n-2}}) = n - 3 + \deg(r)$, which follows from the definition of $\omega_{r, P_{n-2}}$ and the fact that $\deg(r) < \deg(P_{n-2}) = n - 2$. This contradiction implies r is a constant and hence (4.5) holds, $\gcd(r, r') = 1$, and $\omega_{r, P_{n-2}} = r P'_{n-2}$. Therefore and by (4.6), $P'_{n-2}/(n-2) = \gcd(P_{n-2}, P'_{n-2})$ which is possible if and only if $P_{n-2}(\lambda) = (\lambda - \lambda_0)^{n-2}$ for some $\lambda_0 \in \mathbb{C}$. This implies (4.4). \square

5. A criterion when only a 1-step reduction exists

In this section we characterize the pairs $\{P_n, P_{n-2}\}$ which admit only 1-step reductions, that is, have at least one 1-step reduction and are such that all its 1-step reductions cannot be reduced any further.

Proposition 5.1. *The pair $\{P_n, P_{n-2}\}$ satisfying (3.1) admits only 1-step reductions if and only if the following assumptions hold:*

$$n > 6; \quad (5.1)$$

$$P_{n-2}(\lambda) = (\lambda - \lambda_0)^{n-2} + a_0 \quad \text{for some } \lambda_0, a_0 \neq 0; \quad (5.2)$$

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)P_{n-2}(\lambda) + c \quad \text{for some } c \neq 0. \quad (5.3)$$

Proof. Assume (5.1)–(5.3) hold. By Proposition 4.4, the pair $\{P_n, P_{n-2}\}$ admits a reduction. Set $r := P_n \pmod{P_{n-2}} \equiv c$. By Proposition 2.1, the function

$$[r, P_{n-2}]_\gamma(\lambda) = c - \gamma((\lambda - \lambda_0)^{n-2} + a_0)$$

has a zero of multiplicity ≥ 2 for some $\gamma \neq 0$. This is only possible if $\gamma = c/a_0$ and then $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, P_{n-4}\}$ with $P_{n-4}(\lambda) = (\lambda - \lambda_0)^{n-4}$ is the unique 1-step reduction. Again by Proposition 4.4, the pair $\{P_{n-2}, P_{n-4}\}$ has no reductions.

Conversely, assume $\{P_n, P_{n-2}\}$ admits only 1-step reductions. Let $\{P_{n-2}, P_{n-4}\}$ be such a reduction. Then since it cannot be reduced further and by Proposition 4.4, we have $n > 6$, that is, (5.1) holds, and there are complex numbers a_1, b_1, c_1 , and λ_0 such that

$$\begin{aligned} P_{n-4}(\lambda) &= (\lambda - \lambda_0)^{n-4}, \\ P_{n-2}(\lambda) &= (\lambda^2 + a_1\lambda + b_1)P_{n-4}(\lambda) + c_1, \quad c_1 \neq 0. \end{aligned} \quad (5.4)$$

By Proposition 3.1(i), these numbers are unique and without loss of generality we take $\lambda_0 = 0$. Then

$$\begin{aligned} P_n(\lambda) &= (\lambda^2 + a\lambda + b)[(\lambda^2 + a_1\lambda + b_1)\lambda^{n-4} + c_1] - (c\lambda + d)^2\lambda^{n-4}, \\ (c, d) &\neq (0, 0), \end{aligned}$$

and in this representation, by Proposition 3.1(iii), the numbers $a, b, \pm c$, and $\pm d$ are unique also. If $cd \neq 0$, then $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, \tilde{P}_{n-4}\}$ with

$$\tilde{P}_{n-4}(\lambda) = \frac{1}{c^2}(c\lambda + d)^2\lambda^{n-6}$$

is a reduction which, according to Proposition 4.4, can be reduced further. By assumption, this cannot be the case. Hence $cd = 0$, which, because $(c, d) \neq (0, 0)$, implies that either $c = 0$ or $d = 0$. We consider two cases: (I) $c = 0$ and (II) $d = 0$. Set $r := P_n \pmod{P_{n-4}}$.

(I) We show $c = 0$ is impossible. Assume $c = 0$. Then $d \neq 0$,

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)[(\lambda^2 + a_1\lambda + b_1)\lambda^{n-4} + c_1] - d^2\lambda^{n-4}, \quad (5.5)$$

and $r(\lambda) = -d^2\lambda^{n-4}$. We claim $\deg(f_{r, P_{n-2}}) > 2$. To show this, we use

$$\omega_{r, P_{n-2}} = rP'_{n-2} - r'P_{n-2} = \gcd(r, r') \gcd(P_{n-2}, P'_{n-2}) f_{r, P_{n-2}}. \quad (5.6)$$

The first equality in (5.6) implies

$$\omega_{r, P_{n-2}}(\lambda) = -d^2\lambda^{n-5} [\lambda P'_{n-2}(\lambda) - (n-4)P_{n-2}(\lambda)]$$

and hence $\deg(\omega_{r, P_{n-2}}) = 2n - 7$. On the other hand, $\gcd(r, r')(\lambda) = \lambda^{n-5}$ and $\deg(\gcd(P_{n-2}, P'_{n-2})) \leq 2$, which follows from $P_{n-2}(0) \neq 0$ and the fact that P'_{n-2} has at most two nonzero zeros counting multiplicity. Therefore, by the second equality in (5.6), we have

$$\deg(f_{r, P_{n-2}}) \geq 2n - 7 - (n - 5 + 2) = n - 4 > 2,$$

which proves the claim. The claim and Lemma 4.1 imply there is a nonzero γ such that $[r, P_{n-2}]_\gamma = r - \gamma P_{n-2}$ has a zero of multiplicity ≥ 2 . According to Proposition 2.1 and (2.6), there is a reduction $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, \tilde{P}_{n-4}\}$ such that

$$P_n(\lambda) = (\lambda^2 + a\lambda + \tilde{b})P_{n-2}(\lambda) - (\tilde{c}\lambda + \tilde{d})^2 \tilde{P}_{n-4}(\lambda), \quad (5.7)$$

where $\tilde{c}^2 = \gamma \neq 0$. If $\tilde{P}_{n-4} = P_{n-4}$, then, by Proposition 3.1(iii), the two representations (5.5) and (5.7) coincide, but this cannot hold because $(\tilde{c}, \tilde{d}) \neq \pm(0, d)$. Hence $\tilde{P}_{n-4} \neq P_{n-4}$ and since $\{P_{n-2}, \tilde{P}_{n-4}\}$ admits no reductions, we have, by Proposition 4.4,

$$\begin{aligned} \tilde{P}_{n-4}(\lambda) &= (\lambda - \tilde{\lambda}_0)^{n-4} \quad \text{for some } \tilde{\lambda}_0 \neq 0, \\ P_{n-2}(\lambda) &= (\lambda^2 + \tilde{a}_1\lambda + \tilde{b}_1)(\lambda - \tilde{\lambda}_0)^{n-4} + \tilde{c}_1 \quad \text{for some } \tilde{c}_1 \neq 0. \end{aligned} \quad (5.8)$$

The two representations (5.4) and (5.8) are not compatible, since $n > 6$. Hence the equality $c = 0$ is impossible.

(II) $d = 0$. Then $c \neq 0$,

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)[(\lambda^2 + a_1\lambda + b_1)\lambda^{n-4} + c_1] - c^2\lambda^{n-2}, \quad (5.9)$$

and

$$r(\lambda) = -c^2\lambda^{n-2} + c^2P_{n-2}(\lambda) = c^2[(a_1\lambda + b_1)\lambda^{n-4} + c_1].$$

We claim $a_1 = 0$. Assume $a_1 \neq 0$. From $\deg(\omega_{r, P_{n-2}}) = 2n - 6$, $\deg(\gcd(r, r')) \leq 1$, $\deg(\gcd(P_{n-2}, P'_{n-2})) \leq 2$, and (5.6) we obtain

$$\deg(f_{r, P_{n-2}}) \geq 2n - 6 - (1 + 2) = 2n - 9 > 3.$$

On account of (3.1) we have $\text{nul}(r) \cap \text{nul}(P_{n-2}) = \emptyset$. Thus we may apply Lemma 4.1. We consider two cases (A) and (B) and in each of them we obtain a contradiction.

(A) There exists a complex number $\mu \neq 0$ such that $f_{r, P_{n-2}}(\mu) = 0$. By Lemma 4.1, $P_{n-2}(\mu) \neq 0$ and for some polynomial \tilde{P}_{n-4} ,

$$\begin{aligned} [r, P_{n-2}]_\gamma(\lambda) &= r(\lambda) - \gamma P_{n-2}(\lambda) = \gamma(\lambda - \mu)^2 \tilde{P}_{n-4}(\lambda), \\ \gamma &= \frac{r(\mu)}{P_{n-2}(\mu)} \neq 0. \end{aligned}$$

Hence

$$P_n(\lambda) = (\lambda^2 + a\lambda + \tilde{b})P_{n-2}(\lambda) - (\sqrt{\gamma}\lambda - \sqrt{\gamma}\mu)^2 \tilde{P}_{n-4}(\lambda). \quad (5.10)$$

Since $\sqrt{\gamma}\mu \neq 0$, the two representations (5.10) and (5.9) are different, which is in contradiction with Lemma 3.1(iii).

(B) $f_{r,P_{n-2}}(\lambda) = \alpha\lambda^k$, where $\alpha \neq 0$. We determine $k = \deg(f_{r,P_{n-2}})$. Since

$$\begin{aligned} \omega_{r,P_{n-2}}(\lambda) &= (-c^2\lambda^{n-2} + c^2P_{n-2}(\lambda))P'_{n-2}(\lambda) \\ &\quad - (-c^2\lambda^{n-2} + c^2P_{n-2}(\lambda))'P_{n-2}(\lambda) \\ &= -c^2\lambda^{n-3}[\lambda P'_{n-2}(\lambda) - (n-2)P_{n-2}(\lambda)] \end{aligned}$$

and

$$0 \notin \text{nul}(\lambda P'_{n-2} - (n-2)P_{n-2}) \cup \text{nul}(P_{n-2}) \cup \text{nul}(r),$$

we have $k = \deg f_{r,P_{n-2}} = n-3$. But this implies $n-3 \leq 2n-9$, that is, $n \leq 6$, contradicting the earlier conclusion (5.1).

Thus the claim has been proved: $a_1 = 0$. If $b_1 \neq 0$, then $\deg(r) = n-4$ and $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, \tilde{P}_{n-4}\}$ with $\tilde{P}_{n-4} = r/c^2$ is a 1-step reduction which, by Proposition 4.4, can be reduced further, contrary to our assumption. So, also $b_1 = 0$ and $r \equiv c_1c^2 \neq 0$. It follows that (5.2) and (5.3) hold with $\lambda_0 = 0$ and with c replaced by c_1c^2 . \square

6. The main theorem

Theorem 6.1. *The inverse problem for the pair $\{P_n, P_{n-2}\}$ satisfying (3.1) has no solutions if and only if*

$$P_{n-2}(\lambda) = (\lambda - \lambda_0)^{n-2} + a_0 \quad \text{for some } \lambda_0, a_0 \in \mathbb{C}; \quad (6.1)$$

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)[(\lambda - \lambda_0)^{n-2} + a_0] + c \quad \text{for some } c \neq 0; \quad (6.2)$$

$$n > 6 \quad \text{or} \quad n = 5, 6 \quad \text{with } a_0 = 0. \quad (6.3)$$

Proof. If (6.1)–(6.3) hold, then the problem has no solutions by Propositions 4.4 and 5.1.

As to the converse, we first consider some values of n and assume that the inverse problem has no solution.

$n = 4$: By Corollary 4.3, the inverse problem is solvable. Hence $n > 4$.

$n = 5$: If a reduction $\{P_5, P_3\} \rightarrow \{P_3, P_1\}$ exists, the inverse problem is solvable. Thus a 1-step reduction does not exist, hence, according to Proposition 4.4, (6.1)–(6.3) are valid.

$n = 6$: If a reduction $\{P_6, P_4\} \rightarrow \{P_4, P_2\}$, exists, then, by Corollary 4.3, the inverse problem is solvable. Hence $\{P_6, P_4\}$ has no reduction. By Proposition 4.4, (6.1)–(6.3) hold.

$n = 7$: If a reduction $\{P_7, P_5\} \rightarrow \{P_5, P_3\} \rightarrow \{P_3, P_1\}$ exists, the problem is solvable.

Hence $\{P_7, P_5\}$ has at most a 1-step reduction. Now the conditions (6.1)–(6.3) hold with $a_0 = 0$ because of Proposition 4.4 and with $a_0 \neq 0$ because of Proposition 5.1.

$n = 8$: Here, because of Corollary 4.3, $\{P_8, P_6\}$ has at most a 1-step reduction. The conditions (6.1)–(6.3) hold for the same reasons as in the previous case.

From now on we assume $n \geq 9$. We claim that if the conditions (6.1)–(6.2) are not simultaneously satisfied, then $\{P_n, P_{n-2}\}$ admits a 3-step reduction, that is, there is a sequence

$$\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, P_{n-4}\} \rightarrow \{P_{n-4}, P_{n-6}\} \rightarrow \{P_{n-6}, P_{n-8}\},$$

where each subsequent pair is a reduction of previous one. The claim implies the theorem. Indeed, if $\{P_n, P_{n-2}\}$ admits a 3-step reduction, $\{P_{n-2}, P_{n-4}\}$ admits a 2-step reduction, so, by Propositions 4.4 and 5.1, the conditions (6.1)–(6.3) do not hold simultaneously, and hence $\{P_{n-2}, P_{n-4}\}$ has a 3-step reduction, etc.

Proof of the claim. Suppose the claim is not true. Then every 2-step reduction

$$\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, P_{n-4}\} \rightarrow \{P_{n-4}, P_{n-6}\}$$

cannot be reduced further, that is, $\{P_{n-2}, P_{n-4}\}$ has only 1-step reductions. By Proposition 5.1, there are complex numbers λ_0 , which, without loss of generality, we assume to be equal to 0, a_0, a_1, b_1 , and c_1 such that

$$P_{n-4}(\lambda) = \lambda^{n-4} + a_0, \quad a_0 \neq 0, \quad (6.4)$$

$$P_{n-2}(\lambda) = (\lambda^2 + a_1\lambda + b_1)(\lambda^{n-4} + a_0) + c_1, \quad c_1 \neq 0. \quad (6.5)$$

Then P_n has the representation

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)((\lambda^2 + a_1\lambda + b_1)(\lambda^{n-2} + a_0) + c_1) - (c\lambda + d)^2(\lambda^{n-4} + a_0), \quad (c, d) \neq (0, 0), \quad (6.6)$$

where, by Lemma 3.1, the complex numbers $a, b, \pm c$, and $\pm d$ are unique. Our aim is to show that another reduction exists which leads to a contradiction with Lemma 3.1 or to obtain other contradictions, which proves the claim. Some of the arguments and calculations are similar to the ones given in the proof of Proposition 5.1. We consider two cases: (I) $c = 0$ in (6.6), which we show to be impossible, and (II) $c \neq 0$. We set $r = P_n \pmod{P_{n-2}}$.

(I) $c = 0$. Then $\deg(f_{r, P_{n-2}}) \geq n - 4 > 0$. This follows from (5.6) where

- (i) $\gcd(r, r') \equiv 1$ because $r(\lambda) = -d^2(\lambda^{n-4} + a_0)$ and $a_0 \neq 0$,
- (ii) $\deg(\gcd(P_{n-2}, P'_{n-2})) \leq n - 3$, since $\deg(P_{n-2}) = n - 2$, and
- (iii) $\deg(\omega_{r, P_{n-2}}) = 2n - 7$.

By Lemma 4.1, Proposition 2.1, and (2.6), there is a reduction $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, \tilde{P}_{n-4}\}$ such that P_n has the representation

$$P_n(\lambda) = (\lambda^2 + a\lambda + \tilde{b})P_{n-2}(\lambda) - (\tilde{c}\lambda + \tilde{d})^2\tilde{P}_{n-4}(\lambda), \quad \tilde{c} \neq 0. \quad (6.7)$$

If $\tilde{P}_{n-4} = P_{n-4}$ then, by Proposition 3.1(iii), the representations (6.7) and (6.6) with $c = 0$ coincide which cannot be true since $(0, d) \neq \pm(\tilde{c}, \tilde{d})$. Hence $\tilde{P}_{n-4} \neq P_{n-4}$. By assumption, $\{P_{n-2}, \tilde{P}_{n-4}\}$ admits only 1-step reductions and therefore

$$\begin{aligned}\tilde{P}_{n-4}(\lambda) &= (\lambda - \tilde{\lambda}_0)^{n-4} + a_0, \quad \tilde{\lambda}_0 \neq 0, \quad a_0 \neq 0, \\ P_{n-2}(\lambda) &= (\lambda^2 + \tilde{a}_1\lambda + \tilde{b}_1)((\lambda - \tilde{\lambda}_0)^{n-4} + a_0) + \tilde{c}_1.\end{aligned}\quad (6.8)$$

For $n \geq 8$ the representations (6.8) and (6.5) of P_{n-2} are not compatible. Hence the case $c = 0$ is not possible.

(II) $c \neq 0$ in (6.6). Set $\mu = -d/c$. Then

$$\begin{aligned}r(\lambda) &= -c^2[(\lambda - \mu)^2(\lambda^{n-4} + a_0) - P_{n-2}(\lambda)] \\ &= -c^2[((\lambda^2 + a_1\lambda + b_1) - (\lambda - \mu)^2)(\lambda^{n-4} + a_0) + c_1] \\ &= -c^2[(a_1 + 2\mu)\lambda + (b_1 - \mu^2)](\lambda^{n-4} + a_0) + c_1.\end{aligned}\quad (6.9)$$

We consider three cases:

- (A) $a_1 + 2\mu \neq 0$;
- (B) $a_1 + 2\mu = 0$, $b_1 - \mu^2 \neq 0$;
- (C) $a_1 + 2\mu = 0$, $b_1 - \mu^2 = 0$.

(A) From

$$\begin{aligned}\omega_{r, P_{n-2}}(\lambda) &= -c^2[(\lambda - \mu)^2(\lambda^{n-4} + a_0)P'_{n-2}(\lambda) \\ &\quad - \{2(\lambda - \mu)(\lambda^{n-4} + a_0) + (n-4)(\lambda - \mu)^2\lambda^{n-5}\}P_{n-2}(\lambda)]\end{aligned}$$

we deduce that if μ is a zero of $\omega_{r, P_{n-2}}$, then its multiplicity is ≤ 2 . Indeed, suppose μ is a zero of multiplicity ≥ 3 . Then $a_0 = -\mu^{n-4}$ and μ is a zero of

$$q(\lambda) := 2(\lambda^{n-4} - \mu^{n-4}) + (n-4)(\lambda - \mu)\lambda^{n-5}$$

of multiplicity ≥ 2 . Therefore $\mu \in \text{nul}(q')$ and this implies $\mu = 0$. Hence $a_0 = 0$, and we have obtained a contradiction. Now from the first equality in (5.6) and since $\deg(r) = n-3$ and $\deg(P_{n-2}) = n-2$ with $n \geq 9$, we see that each of the zero sets $\text{nul}(r)$ and $\text{nul}(P_{n-2})$ contain more than one number. Hence

$$\deg(\gcd(r, r')) \leq n-5, \quad \deg(\gcd(P_{n-2}, P'_{n-2})) \leq n-4.$$

Moreover, $\deg(\omega_{r, P_{n-2}}) = 2n-6$. Using (5.6) again, we obtain

$$\deg f_{r, P_{n-2}} \geq (2n-6) - (2n-9) = 3.$$

As for $\omega_{r, P_{n-2}}$, if μ is a zero of $f_{r, P_{n-2}}$, then its multiplicity is ≤ 2 . Hence a complex number $\nu \in \text{nul}(f_{r, P_{n-2}})$ exists with $\nu \neq \mu$. By the same reasoning as at the end of case (I) above, this ν generates a reduction $\{P_n, P_{n-2}\} \rightarrow \{P_{n-2}, \tilde{P}_{n-4}\}$ such that P_n has the representation

$$P_n(\lambda) = (\lambda^2 + a\lambda + \tilde{b})P_{n-2}(\lambda) - (\tilde{c}\lambda + \tilde{d})^2\tilde{P}_{n-4}(\lambda), \quad \nu = -\tilde{d}/\tilde{c}, \quad (6.10)$$

which differs from the representation (6.6) and contradicts Proposition 3.1(iii).

(B) Then $\deg(r) = n - 4$, that is, (2.5) holds. By the remark following Proposition 2.1, P_n has the representation (2.4) with $c = 0$ which differs from the representation (6.6). This contradicts Proposition 3.1(iii).

(C) Now $r \equiv -c^2 c_1$ and $\omega_{r, P_{n-2}}(\lambda) = -c^2 c_1 P'_{n-2}(\lambda)$. Hence $\text{nul}(\omega_{r, P_{n-2}}) = \text{nul}(P'_{n-2})$, $\deg(\gcd(r, r')) = 0$, and

$$f_{r, P_{n-2}} = \frac{-c^2 c_1 P'_{n-2}}{\gcd(P_{n-2}, P'_{n-2})}.$$

Since in this case

$$P_{n-2}(\lambda) = (\lambda - \mu)^2(\lambda^{n-4} + a_0) + c_1,$$

we have

$$\begin{aligned} P'_{n-2}(\lambda) &= (\lambda - \mu)[2(\lambda^{n-4} + a_0) + (n-4)(\lambda - \mu)\lambda^{n-5}] \\ &= (\lambda - \mu)[(n-2)\lambda^{n-4} - (n-4)\mu\lambda^{n-5} + 2a_0]. \end{aligned} \quad (6.11)$$

Denote the multiplicity of μ as a zero of P'_{n-2} by m . It can be checked directly that $m \leq 2$. Since $\mu \notin \text{nul}(P_{n-2})$, we have $\mu \in \text{nul}(f_{r, P_{n-2}})$. If there is $\nu \in \text{nul}(f_{r, P_{n-2}})$ with $\nu \neq \mu$, then using arguments as in previous cases we have two representations of P_n , namely

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)P_{n-2}(\lambda) - (c\lambda + d)^2 P_{n-4}(\lambda), \quad \mu = -d/c,$$

which is (6.6), and

$$P_n(\lambda) = (\lambda^2 + a\lambda + b)P_{n-2}(\lambda) - (c_v\lambda + d_v)^2 P_{n-4}^v(\lambda), \quad \nu = -d_v/c_v.$$

The equality $P_{n-4} = P_{n-4}^v$ contradicts Proposition 3.1(iii), therefore $P_{n-4} \neq P_{n-4}^v$. As $n \geq 9$, the inequality that P_{n-4}^v cannot be of the form (4.4) or (5.2) and hence the pair $\{P_{n-2}, P_{n-4}^v\}$ admits a 2-step reduction. This contradicts our assumption, and we conclude $\text{nul}(f_{r, P_{n-2}}) = \{\mu\}$ and $\deg(f_{r, P_{n-2}}) = m$. Moreover, the $n - 3 - m$ zeros of P'_{n-2} which are not equal to μ , are also zeros of P_{n-2} . We consider two cases $m = 2$ and $m = 1$ and show by counting zeros that $n \leq 8$, which contradicts our assumption $n \geq 9$.

First, we show that the zeros of P'_{n-2} have multiplicity ≤ 2 and that there is at most one of multiplicity $= 2$. From (6.11) it follows that μ is a zero of P'_{n-2} of multiplicity 2 (that is, $m = 2$) if and only if $a_0 = -\mu^{n-4}$. Let $\varphi \neq \mu$ be a zero of P'_{n-2} of multiplicity ≥ 2 . Then it is a zero of the polynomial

$$F(\lambda) := (n-2)\lambda^{n-4} - (n-4)\mu\lambda^{n-5} + 2a_0$$

and its derivative

$$F'(\lambda) = (n-4)[(n-2)\lambda - (n-5)\mu]\lambda^{n-6}.$$

It follows that $\varphi := (n-5)\mu/(n-2)$ is the only a candidate to be a zero of F of multiplicity ≥ 2 and if it is, then its multiplicity equals 2. One can check directly that if μ is a zero of P'_{n-2} of multiplicity 2, then φ does not have this property. Hence either μ is a zero of multiplicity 2 or φ is a zero of multiplicity 2, but not both at the same time.

Case $m = 2$. The $n - 5$ zeros of P'_{n-2} which are not equal to μ have multiplicity 1 and are also zeros of P_{n-2} . Hence $n - 2 \geq 2(n - 5)$, that is, $n \leq 8$.

Case $m = 1$. If φ is a zero of P'_{n-2} of multiplicity 2, then we have $n - 2 \geq 2(n - 6) + 3$, that is, $n \leq 7$. If P'_{n-2} only has zeros of multiplicity 1, then we have $n - 2 \geq 2(n - 4)$, that is, $n \leq 6$. \square

We conclude with an example which shows that even in the case when the zeros of P_{2n} and P_{2n-2} have multiplicity 1 and their zero sets are disjoint, it is possible that the inverse spectral problem has no solutions.

Example 6.2. Let $n = 4$ and consider $P_8(\lambda) = \lambda^8 - \lambda^2 + 1$, $P_6(\lambda) = \lambda^6 - 1$. Note that these polynomials have simple zeros and the zero sets are disjoint. From Theorem 5.1 it follows that the pair $\{P_8, P_6\}$ admits only 1-step reductions. This can also be seen directly:

$$P_8(\lambda) = \lambda^2 P_6(\lambda) + 1 = (\lambda^2 + \gamma) P_6(\lambda) + 1 - \gamma(\lambda^6 - 1).$$

The second summand on the right-hand side has a zero of multiplicity ≥ 2 if and only if $\gamma = -1$. In this case we have necessarily $P_4(\lambda) = \lambda^4$,

$$P_8(\lambda) = (\lambda^2 - 1) P_6(\lambda) - \lambda^2 P_4(\lambda),$$

and

$$P_6(\lambda) = \lambda^2 P_4(\lambda) - 1 = (\lambda^2 + \sigma) P_4(\lambda) - (1 + \sigma \lambda^4).$$

Now the second summand on the right-hand side of the last equality does not have a zero of multiplicity ≥ 2 for any $\sigma \in \mathbb{C}$. Hence $\{P_6, P_4\}$ cannot be reduced further. It follows that the inverse spectral problem for $\{P_8, P_6\}$ does not have a solution.

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